ON THE STABILITY OF PERMANENT ROTATIONS OF A RIGID BODY WITH A FIXED POINT UNDER THE ACTION OF A NEWTONIAN CENTRAL FORCE FIELD

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Reference [1] contains a study of the problem of motion of a rigid body with one fixed point under the action of a Newtonian central force field.

The distance R from the center of forces to the fixed point is assumed to be much larger as compared to the dimensions of the rigid body. Thus, in a stationary reference system with the z-axis directed from the center of forces to the fixed point and the x- and y-axes completing a right-handed rectangular system, the projections F_x, F_y, F_z of the forces acting on an element of the rigid body will be to the second order of small quantities

$$F_x = -\frac{g}{R}\,dmx, \qquad F_y = -\frac{g}{R}\,dmy, \qquad F_z = -g\,dm + \frac{2g}{R}\,dmz$$

Here g is the acceleration due to gravity at a distance R from the center of attraction and x, y, z are the coordinates of the element.

Since the forces are potential and stationary, and the ideal constraints do not contain time explicitly , the system of the equation of motion leads to an energy integral

$$H = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + Mg(x_0'\gamma_1 + y_0'\gamma_2 + z_0'\gamma_3) + \frac{3g}{2R}(A\gamma_1^2 B\gamma_2^2 + C\gamma_3^2)$$

Here A, B, C are the principal moments of inertia of the rigid body, x', y', z' are axes aligned along the principal axes of the ellipsoid of inertia, x_0', y_0', z_0' are the coordinates of the mass center of the rigid body, γ_1, γ_2, γ_3 are the direction cosines of the z-axis in the fixed system, p, q, r are the projections of the absolute angular velocity ω on the axes of the moving system. If the generalized coordinates are chosen to be ϕ, ψ, θ, the Eulerian angles of spin, precession and mutation, and the well-known formulas

$$p = \dot\psi \sin\theta \sin\varphi + \dot\theta \cos\varphi, \quad q = \dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi, \quad r = \dot\psi \cos\theta + \dot\varphi$$
$$\gamma_1 = \sin\theta \cos\varphi, \quad \gamma_2 = \sin\theta \sin\varphi, \quad \gamma_3 = \cos\theta$$

are applied, then one can write the expression for the total energy of the system in the following form

$$H = \frac{1}{2}[A(\dot\psi \sin\theta \sin\varphi + \theta \cos\varphi)^2 + B(\dot\psi \sin\theta \cos\varphi - \theta \sin\varphi)^2 + C(\dot\psi \cos\theta + \dot\varphi)^2] + \\ + Mg(x_0' \sin\theta \sin\varphi + y_0' \sin\theta \cos\varphi + z_0' \cos\theta) + \\ + \frac{3g}{2R}(A \sin^2\theta \sin^2\varphi + B \sin^2\theta \cos^2\varphi + C \cos^2\theta)$$

Since ψ is a cyclic coordinate it corresponds to the integral

$$\frac{\partial L}{\partial \dot\psi} = P = A(\dot\psi \sin\theta \sin\varphi + \dot\theta \cos\varphi) \sin\theta \sin\varphi + B(\dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi) \sin\theta \cos\varphi + \\ + B(\dot\psi \cos\theta + \dot\varphi) \cos\theta = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = \text{const}$$

reflecting the constancy of the projection of the kinetic moment upon the z-axis.

If $\dot\phi = \dot\theta = 0$, $\phi = \phi_0$, $\theta = \theta_0$, $\dot\psi = \dot\psi_0$ are the solutions of the equations

$$\frac{\partial H}{\partial \dot\psi} = B_0, \qquad \frac{\partial H}{\partial \dot\varphi} = \frac{\partial H}{\partial \dot\theta} = \frac{\partial H}{\partial \varphi} = \frac{\partial H}{\partial \theta} = 0$$

then, as it is well-known, the equations of motion lead to a particular solution

$$\dot\varphi = \dot\theta = 0, \quad \theta = \theta_0, \quad \varphi = \varphi_0, \quad \dot\psi = \dot\psi_0, \quad \psi = \dot\psi_0 t + \psi_0$$

which is called the permanent rotation.

The analysis of the equations shows that the axes of all permanent rotations coincide with the z-axis and in the moving system are distributed over the cone

$$(B - C) x_0' \gamma_2 \gamma_3 + (C - A) y_0' \gamma_3 \gamma_1 + (A - B) z_0' \gamma_1 \gamma_2 = 0 \tag{1}$$

which is analogous to the Staude cone [2] in the problem of the motion of a heavy rigid body with one fixed point in a uniform gravitational field.

Let us apply the Routh criterion [3] to the analysis. Regardless of the fact that the Routh criterion gives in its formulation sufficient conditions for conditional stability, Liapunov [4] maintained that in relation to non-cyclic coordinates and all velocities the theorem gives sufficient conditions for unconditional stability [5].

Thus, the angles ϕ_0 and θ_0 satisfy equations $\partial H/\partial \phi = 0$, $\partial H/\partial \theta = 0$

constructed on the assumption that $p = B_0$, $\phi = \theta = 0$. These equations are

$$\frac{\sin 2\theta}{m}\left(\frac{3g}{R}-\frac{B_0^2}{K^2}\right)+Mg\,(x_0'\cos\theta\sin\varphi+y_0'\cos\theta\cos\varphi-z_0'\sin\theta)=0$$
$$\frac{(A-B)\sin 2\varphi\sin^2\theta}{2}\left(\frac{3g}{R}-\frac{B_0^2}{K^2}\right)+Mg\,(x_0'\sin\theta\cos\varphi-y_0'\sin\theta\sin\varphi)=0 \qquad (2)$$

where

$$K=A\sin^2\theta\sin^2\varphi+B\sin^2\theta\cos^2\varphi+C\cos^2\theta,\quad m=A\sin^2\varphi+B\cos^2\varphi-C.$$

The equation of the Staude cone in terms of the variables ϕ, θ is obtained by eliminating from these equations the quantities in parentheses.

Note that the equations analogous to equations (2) for a rigid body in a uniform gravitational field are obtained by replacing the quantity $3g/R - p_0{}^2/k^2$ by $(-\,p_0{}^2/k^2)$. Therefore, the regions of possible axes under study will be somewhat wider. Actually, if these quantities are treated in every problem as an arbitrary parameter, then it can be seen that in the problem under study it varies from $3g/R$ to $-\infty$, and in the problem with a uniform field it varies from 0 to $-\infty$.

The variable potential energy of the system is of the form:

$$F=\frac{p_0^2}{2K}+Mg\,(x_0'\sin\theta\sin\varphi+y_0'\sin\theta\cos\varphi+z_0'\cos\theta)+\frac{3g}{2R}K$$

The motion (a) will be stable in relation to ϕ, θ, ψ, $\dot\phi$, $\dot\theta$ if the function

$$\delta^2F=\frac{\partial^2F}{\partial\varphi^2}\delta\varphi^2+2\frac{\partial^2F}{\partial\theta\partial\varphi}\delta\varphi\delta\theta+\frac{\partial^2F}{\partial\theta^2}\delta\theta^2$$

is found to be positive with respect to $\delta\theta$, $\delta\phi$, which will be the case if, and only if

$$\frac{\partial^2F}{\partial\varphi^2}>0,\qquad \frac{\partial^2F}{\partial\varphi^2}\frac{\partial^2F}{\partial\theta^2}-\left(\frac{\partial^2F}{\partial\varphi\partial\theta}\right)^2>0$$

These conditions are

$$m\left(\frac{3g}{R}-\omega^2\right)\cos 2\theta+\frac{\omega^2m^2}{K}\sin^2 2\theta-Mg\,(x_0'\sin\theta\sin\varphi+y_0'\sin\theta\cos\varphi+z_0'\cos\theta)>0$$

$$\left[\left(\frac{3g}{R}-\omega^2\right)\cos 2\theta m+\frac{\omega^2m^2}{K}\sin^2 2\theta-Mg\,(x_0'\sin\theta\sin\varphi+y_0'\sin\theta\cos\varphi+z_0'\cos\theta)\right]\times$$
$$\times\left[(A-B)\left(\frac{3g}{R}-\omega^2\right)\cos 2\varphi\sin^2\theta+\frac{\omega^2\sin^4\theta\sin^2 2\varphi\,(A-B)^2}{K}-Mg\sin\theta\,(x_0'\sin\varphi+\right.$$
$$\left.+\,y_0'\cos\varphi)\right]-\left[\frac{1}{2}\left(\frac{3g}{R}-\omega^2\right)\sin 2\theta\sin 2\varphi\,(A-B)+\omega^2\sin^2 2\varphi\sin 2\theta\sin^2\theta\,\frac{m}{K}-\right.$$
$$\left.-\,Mg\cos\,(x_0'\sin\varphi+y_0'\cos\varphi)\right]>0\qquad\left(\omega^2=\frac{p_0^2}{K^2}\right)$$

Together with equations (2) they single out the stable permanent axes.

The study of the general inequalities is very difficult. However, if $A = B$, and $z_0' = 0$, then for $x_0' > 0$, $y_0' > 0$ and $\sin\theta_0 > 0$ they reduce to the inequalities

$$x_0' > y_0', \quad \left(\frac{\cos 2\theta - \sin^2\theta}{\sin\theta}\right) + \left(\frac{\sqrt{x_0'^2 + y_0'^2}}{\sin\theta\,(A - C)} + \frac{3g}{MR}\right)\frac{\sin^2\theta\,(A - C)^2\sqrt{x_0'^2 + y_0'^2}}{Ax_0'^2 + By_0'^2} > 0$$

The problem of the stability of permanent rotations of a rigid body in a uniform gravitational field was studied in detail by Rumiantsev [6].

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